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Bihamiltonian structures and Stäckel separability

A. Ibort [a,*], F. Magri [b], G. Marmo [c]

[a] Depto. de Matemáticas, Univ. Carlos III de Madrid, Avda. de la Universidad 30, 28911 Leganés, Madrid, Spain

[b] Dipto. di Matematica e Applicazioni, Univ. di Milano Bicocca, Viale Sarca 203, 20100 Milano, Italy

[c] Dipto. di Scienze Fisiche, INFN, Univ. di Napoli, Mostra d'Oltremare, Pad. 19, 80125 Naples, Italy

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Abstract

It is shown that a class of Stäckel separable systems is characterized in terms of a Gel'fand–Zakharevich bihamiltonian structure. This structure arises as an extension of a Poisson–Nijenhuis structure on phase space. It is also shown that the Casimir of the Gel'fand–Zakharevich bihamiltonian structure provides the family of commuting Killing tensors found by Benenti and that, because of Eisenhart's theorem, characterize orthogonal separability. It is also shown that recently found properties of quasi-bihamiltonian systems are natural consequences of the geometry of the extension of the Poisson–Nijenhuis structure.

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1. Introduction

The construction and characterization of separable Hamiltonian systems has an old and fascinating history that goes back to the work of Hamilton and Jacobi [10] and that is gaining a renewed interest nowadays in the realm of integrable systems (see for instance [7,15]).

Early characterizations of separable systems obtained by Stäckel [17], Levi-Civita [16], etc., were brought to maturity by the work of Eisenhart that put the problem in an intrinsic and geometric form. Eisenhart theorem settled the question of the orthogonal separability of geodesical motion in Riemannian manifolds. Orthogonal separable geodesic motions are characterized by the existence of families of quadratic Killing tensors satisfying

* Corresponding author.

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an appropriate set of conditions, and they are necessarily of Stäckel form [6,11,12,22]. These ideas are nicely embraced by the geometric notion of Killing webs discussed in [2,3].

Benenti [1] presented a class of orthogonal separable systems characterized geometrically by a single tensor L satisfying certain conditions. The corresponding family of Killing tensors were obtained from the tensor L by a set of recurrence conditions. These constants of the motion however are not of the type obtained from recursion operators, i.e., traces of powers of a (1,1) tensor. Hence the quadratic constants of the motion provided by Killing tensors for separable systems do not fit exactly into the scheme of Liouville complete integrability and recursion operators developed in recent years (see for instance [8,13,14]).

We will propose in this paper a geometrical setting that gives a coherent geometrical framework for Benenti's results in separability and complete integrability. The main idea is that the natural framework to understand Benenti's construction is that of bihamiltonian systems. The bihamiltonian structure arises not in the phase space of the original system but on an extension of it. The emerging structure is that of a Gel'fand–Zakharevich bihamiltonian manifold [9] whose Casimir polynomial give the sought Killing tensors constructed by Benenti. In this way we show that the natural setting for the description of these systems is not the well known theory of recursion operators but bihamiltonian manifolds of Gel'fand–Zakharevich type. As a by-product of the theory we will show that recent results on quasi-bihamiltonian systems [19] fit nicely into this scheme.

The paper is organized as follows. In Section 2 we will review the theorems by Stäckel, Eisenhart and Benenti. Section 3 will contain the main results of the paper. In particular we describe here the Poisson–Nijenhuis and Gel'fand–Zakharevich structures associated to a Riemannian manifold of Benenti type. Finally, in Section 4, we will describe the converse perspective, from bihamiltonian systems to separability, by means of two examples, stationary reduction of the KdV hierarchy and systems of hydrodynamic type.

2. Stäckel and Benenti systems

2.1. *Stäckel's theorems*

In 1893 Stäckel gave the first characterization of the Riemannian manifolds (Q, g) on which the equations of geodesic motion can be solved by separation of variables in an orthogonal system of coordinates. He made two remarks. First, he noticed [17] that the Hamilton–Jacobi equation associated with the kinetic energy

$$K = \frac{1}{2}\sum_i g^{ii} p_i^2, \tag{2.1}$$

written in a system of orthogonal coordinates q^i is separable if there exists a regular matrix $\varphi = (\varphi_k^l(q^k))$, called a Stäckel matrix, such that,

$$g^{kk} = (\varphi^{-1})_1^k. \tag{2.2}$$

Then, he noticed [18] that the functions,

$$K_l = \sum_{j=1}^{n} (\varphi^{-1})_l^j p_j^2, \tag{2.3}$$

defined by the same matrix φ, pairwise commute with respect to the canonical Poisson bracket. In this way he proved that a geodesic flow on a Riemannian manifold separable in an orthogonal system of coordinates admits a family of $n = \dim Q$ independent commuting integrals of the motion which are quadratic in the momenta p_i's.

2.2. *Eisenhart's theorem*

In 1934 Eisenhart [6] proved that the converse of this result is also true, and gave an intrinsic form to Stäckel theorems by exploiting the concept of Killing tensors. For any symmetric contravariant tensor K of order r on Q let us denote by $\hat{K}$ the homogeneous function of degree r on the momenta variables p_i defined by

$$\hat{K}(q, p) = K^{i_1 \ldots i_r}(q) p_{i_1} \ldots p_{i_r}.$$

If $G = g^{ij} \partial/\partial q^i \otimes \partial/\partial q^j$ is the contravariant metric tensor defined by the metric g, it is usual to say that K is a Killing tensor if

$$\{\hat{G}, \hat{K}\} = 0. \tag{2.4}$$

A tensor L verifying the equation,

$$\{\hat{L}, \hat{G}\} = c\hat{G}, \tag{2.5}$$

is instead called a conformal Killing tensor. A simple example is the tensor $L = K + fG$ obtained by adding to K a non-constant multiple of the metric tensor. In this case the conformal factor c is given by

$$c = g^{ij} p_j \frac{\partial f}{\partial q^i}.$$

By this reason tensors L of this type will be called conformal Killing tensors of gradient type, and the function f will be called the associated potential.

Eisenhart realized that Stäckel's results can be given a coordinate-free form by considering special families of Killing tensors of order 2. He proved that the kinetic energy K has the Stäckel form (2.2) if the metric tensor G has $n - 1$ commuting independent Killing tensors of order 2, $\hat{K}_{(1)} = K_{(1)}^{ij} p_i p_j, \ldots, \hat{K}_{(n-1)} = K_{(n-1)}^{ij} p_i p_j$, admitting a common system of closed eigenforms α_i, i.e.,

$$(K_{(l)} - \rho_{(l)}^i G)\alpha_i = 0, \quad \mathrm{d}\alpha_i = 0.$$

Furthermore, he gave a simple algebraic procedure to reconstruct the Stäckel matrix φ from the metric tensor itself and the eigenvalues $\rho_{(l)}^i$ of its Killing tensors. By this result the theory of separability in orthogonal coordinates was turned towards the study of special families of Killing tensors.

2.3. Benenti's theorem

In 1992 Benenti has shown a simple recurrence procedure to construct a family of Killing tensors obeying the assumptions of Eisenhart's theorem. He considered a special class of Riemannian manifolds (Q, g, L) endowed with a second symmetric tensor field L of type (2,0) such that [1]:

1. The Nijenhuis torsion of L vanishes, $N_L = [L, L] = 0$, where L is here regarded as the (1,1) tensor field defined by raising one index with the metric g.
2. L is a conformal Killing tensor of gradient type.
3. The associated potential f is the trace of L,

$$\{\hat{L}, \hat{G}\} = c\hat{G}, \quad c = p_j g^{ij} \frac{\partial L_k^k}{\partial q^i}. \tag{2.6}$$

Under these conditions he proved [1] that the tensors,

$$K_{(l)} = \sum_{j=0}^{l} (-1)^l \sigma_{l-j}(\lambda_1, \ldots, \lambda_n) L^k \circ G, \quad l = 1, \ldots, n-1, \tag{2.7}$$

where the functions σ_a are the elementary symmetric polynomials of degree a on the eigenvalues of L, are Killing tensors satisfying Eisenhart's theorem. He was then able to conclude that the geodesic flow on Q is separable in orthogonal coordinates, and to construct the separation coordinates from the study of the eigenvectors of L.

2.4. Separable potentials

The previous discussion and theorems were also extended to include mechanical systems with potential terms. The basic result [2,3] is that the Hamiltonian

$$H = \tfrac{1}{2} g^{ij} p_i p_j + V,$$

is separable in a suitable system of orthogonal coordinates iff the metric G has $n-1$ Killing tensors as in Eisenhart's theorem and the potential V verifies the equation,

$$\mathrm{d}(K\mathrm{d}V) = 0$$

for K a Killing tensor with simple eigenvalues. In this case,

$$\mathrm{d}(K_{(l)}\mathrm{d}V) = 0, \quad l = 1, \ldots, n-1,$$

for all the commuting Killing tensors provided by Eisenhart's theorem. This allows to (locally) define the new potentials V_l according to

$$\mathrm{d}V_l = K_{(l)}\mathrm{d}V, \tag{2.8}$$

and to introduce the Hamiltonian functions,

$$H_l = \tfrac{1}{2}\hat{K}_{(l)} + V_l. \tag{2.9}$$

It can be proved that these functions form a family of commuting integrals of the motion for the Hamiltonian H. In this paper we will call Stäckel systems the Hamiltonian systems associated with the functions (2.9) and Benenti systems those corresponding to the Benenti–Killing tensors (2.7). Our aim is to provide a new geometrical characterization of Benenti systems, showing that, in a suitable sense, they are bihamiltonian. We hope that this result could help to shed some light on the still unclear relationship between separable and bihamiltonian systems.

3. The bihamiltonian structure of Benenti systems

3.1. The Poisson–Nijenhuis structure on T^*Q

We shall start the analysis of Benenti systems studying the geometry of the cotangent bundle associated with the manifold (Q, g, L). Our first result is that T^*Q is a bihamiltonian manifold. This means that T^*Q is naturally endowed with a pair of compatible Poisson brackets $\{.,.\}_0$ and $\{.,.\}_1$. That the brackets $\{.,.\}_0$ and $\{.,.\}_1$ are compatible means that the linear combination

$$\{f, g\}_s = \{f, g\}_0 + s\{f, g\}_1,$$

verifies the Jacobi identity for any value of the real parameter s. The first bracket $\{.,.\}_0$ is the canonical Poisson bracket

$$\{f, g\}_0 = \omega_0(X_f, X_g),$$

defined on any cotangent bundle T^*Q by the symplectic 2-form $\omega_0 = -\mathrm{d}\theta_0$. Here $\theta_0 = p_k \mathrm{d}q^k$ is the Liouville 1-form and X_f is the Hamiltonian vector field associated with the scalar function $f : T^*Q \to \mathbf{R}$.

The second Poisson bracket $\{f, g\}_1$ is constructed by using the tensor field L that from now on will be thought as a (1,1) tensor field on Q, i.e., a bundle endomorphism $L : TQ \to TQ$. First we use L to deform the Liouville 1-form according to the simple expression

$$\theta_1 = L^*\theta_0 = L_i^j(q) p_j \mathrm{d}q^i, \tag{3.10}$$

and define the 2-form $\omega_1 = \mathrm{d}\theta_1$. Then we set,

$$\{f, g\}_1 = \omega_1(X_f, X_g), \tag{3.11}$$

where X_f, X_g are again the Hamiltonian vector fields corresponding to the functions f and g with respect to the symplectic structure ω_0. In local coordinates this definition reads:

$$\{q^i, q^j\}_1 = 0, \quad \{q^i, p_j\}_1 = -L_j^i, \quad \{p_i, p_j\}_1 = \left(\frac{\partial L_j^k}{\partial q^i} - \frac{\partial L_i^k}{\partial q^j}\right) p_k, \tag{3.12}$$

and the matrix expression of the Poisson tensor π_1 corresponding to this bracket is given by

$$\pi_1 = \begin{bmatrix} 0 & -L_i^j \\ L_j^i & \left(\frac{\partial L_j^k}{\partial q^i} - \frac{\partial L_i^k}{\partial q^j}\right) p_k \end{bmatrix}.$$

Proposition 1. *Under the condition that the Nijenhuis torsion of L vanishes,* Eq. (3.11) *defines a second Poisson bracket on T^*Q compatible with the canonical one.*

Proof. A simple computation shows that the Jacobi condition for the bracket $\{\{q^i, p_j\}_1, p_k\}_1$, and its cyclic permutations corresponds to the vanishing of the Nijenhuis torsion of L. The Jacobi condition for the bracket $\{\{p_i, p_j\}_1, p_k\}_1$ and its cyclic permutations is a differential consequence of the previous one. Finally, the two Poisson brackets $\{.,.\}_0$ and $\{.,.\}_1$ are compatible since ω_1 is a closed 2-form. □

Since ω_0 is symplectic, there exist a unique tensor field N of type (1,1) on T^*Q such that

$$\omega_1(X, Y) = \omega_0(NX, Y),$$

for any pair of vector fields X, Y on T^*Q. Otherwise $N = \pi_1 \circ \omega_0$ and N is called the Nijenhuis tensor (or recursion operator) associated with the pair of Poisson brackets considered. In local coordinates it is defined by

$$N^*(\mathrm{d}q^k) = L_j^k \mathrm{d}q^j, \quad N^*(\mathrm{d}p_k) = L_k^j \mathrm{d}p_j - p_j \mathrm{d}L_k^j.$$

These formulas show that N is the complete lifting of L to T^*Q. According to the terminology in use in the theory of Poisson manifolds, we can say that the cotangent bundle of (Q, g, L) is a Poisson–Nijenhuis manifold. It is of a special type: the Nijenhuis tensor N preserves the bundle structure T^*Q and is homogeneous of degree 0 on the p coordinates. Correspondingly the Poisson bracket π_1 is homogeneous of degree -1 The Liouville vector field $\Delta = p_i \partial/\partial p_i$ characterizes completely the bundle structure of T^*Q, hence preserving such structure is equivalent to the vanishing of the bracket $[N, \Delta] = 0$ which is another way of expressing that N is homogeneous of degree 0.

Remark. *A remark is in order here. The constants of the motion obtained from the Poisson–Nijenhuis structure on T^*Q do not provide a system of Killing tensors in the sense of Eq. (2.7). In fact, Tr $N^k = 2\,\mathrm{Tr}\, L^k$, hence it would define constants of the motion which are just functions on Q.*

3.2. The Gel'fand–Zakharevich structure on $T^*Q \times \mathbf{R}$

The next step is to "geometrize" (in the sense of the geometry of Poisson manifolds) the Killing tensors (2.7) and the separable potentials (2.8) considered by Benenti. It turns out that T^*Q is too small to do that suitably. We need to enlarge T^*Q and pass to $T^*Q \times \mathbf{R}$ according to a process which is familiar in the framework of Kaluza–Klein theories. This step obliges to leave the theory of Poisson–Nijenhuis manifolds, and to enter into a new field worked out by Gel'fand and Zakharevich [9]. These authors have studied the geometry of an odd-dimensional manifold M endowed with a pair of compatible Poisson tensors P_0, P_1 of maximal rank. Under this assumption they have shown that the Poisson pencil $P_s = P_0 + sP_1$ possesses a Casimir function $\hat{H}(s)$ which is a polynomial in s of degree n,

$$\hat{H}(s) = \hat{H}_0 s^n + \hat{H}_1 s^{n-1} + \cdots + \hat{H}_n,$$

if $\dim M = 2n + 1$. The following properties of the coefficients $\hat{H}_k$ are easily proved:

1. $\hat{H}_0$ is a Casimir of the first Poisson bracket P_0; $\hat{H}_n$ is a Casimir of the second Poisson bracket P_1.
2. The coefficients $\hat{H}_k$ verify the Lenard recursion relations:

$$\{.,\hat{H}_{k+1}\}_0 = \{.,\hat{H}_k\}_1. \tag{3.13}$$

3. The coefficients $\hat{H}_k$ are in involution with respect to both Poisson brackets.

Our strategy will be to construct a suitable extension of the Poisson–Nijenhuis structure on T^*Q to a Gel'fand–Zakharevich structure on $T^*Q \times \mathbf{R}$ in such a way that the coefficients $\hat{H}_k$ of the Casimir function are exactly the Hamiltonian functions considered by Benenti. Extensions of Poisson structures has been considered in various settings (see for instance [4] for the Lie algebraic setting).

We consider the manifold $M = T^*Q \times \mathbf{R}$ with coordinates (q^i, p_i, E) and we define P_0 as the direct product of the canonical Poisson structure on T^*Q and the trivial one in $\mathbf{R}$. Then, E will be the Casimir of the first Poisson structure P_0 on M and T^*Q will be imbedded as a symplectic submanifold of M corresponding to $E = 0$. The Poisson bracket defined by the Poisson tensor P_0 will be denoted again by $\{.,.\}_0$.

The extension of the second Poisson structure will be realized by imposing the first recurrence relation (3.13) of Gel'fand–Zakharevich. We set

$$\hat{H}_0 = E,$$
$$\hat{H}_1 = \tfrac{1}{2} g^{ik} p_i p_k + V + f(q)E, \tag{3.14}$$

where g is the Riemannian metric, V the first separable potential and f is the potential of the conformal Killing tensor L used in Benenti theory. To define the Poisson tensor P_1 on M, we impose the condition,

$$\{., E\}_1 = \{.,\hat{H}_1\}_0. \tag{3.15}$$

We shall denote again the Poisson bracket on M defined by P_1 by $\{.,.\}_1$. The commutation relations for the extension of the Poisson bracket $\{.,.\}_1$ are given by Eq. (3.12) together with

$$\{q^i, E\}_1 = \frac{\partial \hat{H}_1}{\partial p_i}, \quad \{p_i, E\}_1 = -\frac{\partial \hat{H}_1}{\partial q^i}. \tag{3.16}$$

Proposition 2. *The brackets* $\{.,.\}_0$ *and* $\{.,.\}_1$ *endow* M *with the structure of a bihamiltonian manifold of Gel'fand–Zakharevich type if and only if the following conditions are verified:*

1. $d(L^* df) = 0$.
2. $d((L^* - fI)dV) = 0$,

where f is the potential of the torsionless conformal Killing tensor L.

Proof. We should check first the Jacobi condition for $\{.,.\}_1$. Because $\{.,.\}_1$ is an extension of a Poisson bracket on T^*Q we only need to check the Jacobi condition for the following brackets: $\{\{q^i, q^j\}_1, E\}_1$, $\{\{q^i, p_j\}_1, E\}_1$ and $\{\{p_i, p_j\}_1, E\}_1$. Thus for the first one, we easily obtain,

$$\{\{q^i, q^j\}_1, E\}_1 + \text{cyclic} = L^i_k g^{kj} - L^j_k g^{kj} = 0,$$

because L is symmetric. Thus the Jacobi condition for q^i, q^j and E amounts to the orthogonality of the eigenvectors of the tensor L.

The Jacobi identity for q^i, p_j and E demands that L is a conformal Killing tensor of gradient type for the Riemannian metric g with potential the function f entering the definition of the extension $\hat{H}_1$ of the Hamiltonian on T^*Q.

Finally, the Jacobi identity for p_i, p_j and E splits into three conditions. One is a differential consequence of the previous condition; the second imposes the condition $d(L^* df) = 0$ on f. The third one, finally, gives the condition $d((L^* - fI)dV) = 0$ for the potential V.

The compatibility of the two Poisson tensors P_0 and P_1 follows from the compatibility of π_0 and π_1 on T^*Q and Eq. (3.15). □

Conditions (1) and (2) in the proposition above are certainly verified by the choice

$$f = \operatorname{Tr} L,$$

where now (2) becomes $d(K_{(1)} dV) = 0$.

We can therefore summarize the previous discussion by saying that the extended cotangent bundle $T^*Q \times \mathbf{R}$ associated with the manifold (Q, g, L) considered by Benenti are bihamiltonian manifolds of Gel'fand–Zakharevich type.

3.3. The Casimir polynomial

We are now in the proper setting to interpret the relations (2.7) and (2.8) for the Killing tensors $K_{(l)}$ and the separable potentials V_l, respectively. They are an instance of the recursion relation (3.13) on the coefficients of the Casimir function on a bihamiltonian manifold of Gel'fand–Zakharevich type.

We shall expend the rest of this section to prove this statement. We first notice a remarkable homogeneity property of the Poisson tensor P_1. If we assign degree 1 to the momenta p_i and degree 2 to the new coordinate E (this could be made explicit by writing E as p_0^2), and we introduce the dilation vector field Δ_M on M given by

$$\Delta_M = p_i \frac{\partial}{\partial p_i} + 2E \frac{\partial}{\partial E},$$

then, it is easy to show that

$$\mathcal{L}_{\Delta_M} P_1 = -P_1,$$

and the Poisson tensor P_1 is homogeneous of degree -1 on M.

It follows that if $\hat{H}_k$ is quadratic, $\mathcal{L}_{\Delta_M}\hat{H}_k = 2\hat{H}_k$, and $\hat{H}_{k+1}$ verifies the recurrence relation

$$\{., \hat{H}_k\}_1 = \{., \hat{H}_{k+1}\}_0, \tag{3.17}$$

the $\hat{H}_{k+1}$ will also be quadratic on M. Therefore it is natural to look for Gel'fand–Zakharevich Hamiltonians $\hat{H}_k$ having the form

$$\begin{aligned} \hat{H}_0 &= E, \\ \hat{H}_1 &= \tfrac{1}{2}g^{ij}_{(1)}p_i p_j + V_1(q) + f_1(q)E, \\ \hat{H}_2 &= \tfrac{1}{2}g^{ij}_{(2)}p_i p_j + V_2(q) + f_2(q)E, \\ &\vdots \\ \hat{H}_n &= \tfrac{1}{2}g^{ij}_{(n)}p_i p_j + V_n(q) + f_n(q)E. \end{aligned} \tag{3.18}$$

Thus we have shown that the Casimir of the bihamiltonian structure defined on the Gel'fand–Zakharevich extension M defined by a Benenti system (Q, g, L), lies in the family of quadratic functions on M.

3.4. The recurrence relations

We shall compute now the family of quadratic constants of the motion obtained by this method, Eqs. (3.18).

Proposition 3. *The Gel'fand–Zakharevich recurrence relations $P_0(\mathrm{d}\hat{H}_{a+1}) = P_1(\mathrm{d}\hat{H}_a)$, $a = 0, \dots, n-1$ for the family of Hamiltonians (3.18) are equivalent to the set of equations:*

$$\mathrm{d}f_{a+1} = -L^*\mathrm{d}f_a + f_a\mathrm{d}f_1, \tag{3.19}$$

$$\mathrm{d}V_{a+1} = -L^*\mathrm{d}V_a + f_a\mathrm{d}V_1, \tag{3.20}$$

$$G_{(a+1)} = -LG_{(a)} + f_a G_{(1)}, \tag{3.21}$$

where $G_{(a)}$ is the contravariant tensor whose components are $g^{ij}_{(a)}$.

Proof. Computing the different components of the vector fields $X_{a+1} = P_0(\mathrm{d}\hat{H}_{a+1}) = P_1(\mathrm{d}\hat{H}_a)$ we will obtain the recurrence relations:

$$\frac{\partial \hat{H}_{a+1}}{\partial p_i} = -L^j_i \frac{\partial \hat{H}_a}{\partial p_j} + \frac{\partial \hat{H}_1}{\partial p_i}\frac{\partial \hat{H}_a}{\partial E}, \tag{3.22}$$

$$\frac{\partial \hat{H}_{a+1}}{\partial q^i} = -L^j_i \frac{\partial \hat{H}_a}{\partial q^j} + \left(\frac{\partial L^k_j}{\partial q^i} - \frac{\partial L^k_i}{\partial q^j}\right) p_k \frac{\partial \hat{H}_a}{\partial p_j} - \frac{\partial \hat{H}_1}{\partial q^i}\frac{\partial \hat{H}_a}{\partial E}, \tag{3.23}$$

$$\frac{\partial \hat{H}_1}{\partial q^i}\frac{\partial \hat{H}_a}{\partial p_i} = \frac{\partial \hat{H}_1}{\partial p_i}\frac{\partial \hat{H}_a}{\partial q^i}. \tag{3.24}$$

Particularizing for quadratic Hamiltonians of the form given by Eqs. (3.18), the previous Eqs. (3.22)–(3.24) become the family of relations (3.19)–(3.21), together with the compatibility conditions:

$$G_{(1)}\mathrm{d}f_a = G_{(a)}\mathrm{d}f_1, \tag{3.25}$$

$$G_{(1)}\mathrm{d}V_a = G_{(a)}\mathrm{d}V_1, \tag{3.26}$$

$$G_{(1)}\mathrm{d}G_{(a)} = G_{(a)}\mathrm{d}G_{(1)}, \tag{3.27}$$

$$\mathrm{d}G_{(a+1)} = -L^*\mathrm{d}G_{(a)} - f_a\mathrm{d}G_{(1)} - 2G_{(a)}\mathrm{d}L. \tag{3.28}$$

The consistency of the compatibility conditions (3.25)–(3.27) are easily proved by induction as follows. They are obviously true for $a = 1$. Let us consider now Eq. (3.25) for $a+1$, then we get by using the recurrence relation Eqs. (3.19) and (3.21)

$$\begin{aligned} G_{(1)}\mathrm{d}f_{a+1} - G_{(a+1)}\mathrm{d}f_1 &= G_{(1)}(-L^*\mathrm{d}f_a + f_a\mathrm{d}f_1) - (-L^*G_{(a)} + f_aG_{(1)})\mathrm{d}f_1 \\ &= LG_{(a)}\mathrm{d}f_1 - LG_{(1)}\mathrm{d}f_a = L(G_{(a)}\mathrm{d}f_1 - \mathrm{d}f_aG_{(1)}) = 0. \end{aligned}$$

The compatibility conditions for V_a and $G_{(a)}$ are treated in the same way. The last condition requires a longer (but straightforward) computation which will not be detailed here. □

3.5. Solving the recurrence relations: the Leverrier–Newton method

We are finally left with solving the three recursion relations on the functions f_a, V_a and the Killing tensors $G_{(a)}$ given by Eqs. (3.19)–(3.21). To this end we have to use, once again, the condition that the Nijenhuis torsion of L vanishes. As we have seen before, this condition was crucial to guarantee the existence of a second Poisson structure on T^*Q. Presently, it plays a crucial role in solving the first recurrence relation (3.19) on the functions f_a. We shall show that (3.19) is a generalized form of the Leverrier–Newton recurrence formula for computing the characteristic polynomial of a given matrix [21].

Let L be a diagonalizable endomorphism of the tangent bundle of the manifold Q possessing a real spectrum given by the eigenfunctions $\lambda_1, \dots, \lambda_n$. We define the canonical symmetric 1-forms $\alpha_k = \mathrm{d}\sigma_k$, where the functions $\sigma_k(\lambda_1, \dots, \lambda_n)$ are the canonical symmetric functions of the eigenfunctions λ_a.

If the tensor L has vanishing Nijenhuis torsion, in a neighborhood of a point where the eigenfunctions λ_a are independent, it can be brought into the normal form

$$L = \lambda_a \frac{\partial}{\partial \lambda_a} \otimes \mathrm{d}\lambda_a.$$

If we denote by $p_k = \operatorname{Tr} L^k$ the traces of powers of the tensor L, then the Leverrier method for finding the characteristic polynomial $p_L(z) = z^n + a_1\lambda^{n-1} + \cdots + a_{n-1}z + a_n$ of L, allows to compute recursively its coefficients a_k by means of Newton's formula,

$$\begin{aligned} &a_1 = -p_1, \\ &2a_2 = -(p_2 + a_1p_1), \dots, ka_k = -(p_k + a_1p_{k-1} + \cdots + a_{k-1}p_1), \dots \end{aligned} \tag{3.29}$$

but, a simple computation shows that

$$L^*(\alpha_1) = \tfrac{1}{2}\mathrm{d}\,\mathrm{Tr}\,L^2, \quad (L^*)^k(\alpha_1) = \frac{1}{k}\mathrm{d}\,\mathrm{Tr}(L^*)^k, \dots ,$$

hence, because $a_k = (-1)^k\sigma_k$, we have

$$\alpha_1 = \mathrm{d}\sigma_1 = -\mathrm{d}a_1 = \mathrm{d}p_1 = \mathrm{d}\,\mathrm{Tr}\,L,$$

$$\alpha_2 = \mathrm{d}\sigma_2 = \mathrm{d}a_2 = -\tfrac{1}{2}\mathrm{d}(p_2 + a_1 p_1) = -\tfrac{1}{2}(\mathrm{d}\,\mathrm{Tr}\,L^2 - 2p_1\mathrm{d}\,\mathrm{Tr}\,L) = -L^*\alpha_1 + \sigma_1\alpha_1,$$

and, in general we have,

$$\alpha_k = -L^*\alpha_{k-1} + \sigma_{k-1}\alpha_1. \tag{3.30}$$

Formula (3.30) allows to compute iteratively α_k, to obtain,

$$(-1)^a\alpha_{a+1} = [(L^*)^a - \sigma_1(L^*)^{a-1} + \sigma_2(L^*)^{a-2} + \cdots + (-1)^a\sigma_a]\alpha_1, \tag{3.31}$$

which is the form that Newton's formula (3.29) takes now.

Therefore, the functions σ_a solve exactly the first recurrence relation (3.19). So we can set,

$$f_a = \sigma_a.$$

This is not the most general solution of this equation; however, it is the solution implicitly chosen by Benenti. Notice that $\mathrm{d}(\mathrm{d}f_a) = 0$, implies that for $a = 1$, $\mathrm{d}(L^*\mathrm{d}f_1) = 0$. Then if $\mathrm{d}f_1 = \Sigma_i a_i \mathrm{d}\lambda_i$, $L^*(\mathrm{d}f_1) = \Sigma_i a_i\lambda_i\mathrm{d}\lambda_i$, and $\partial a_i/\partial\lambda_j = 0, i \neq j$. All solutions of Eq. (3.19) can be computed iteratively from $f_1 = A_1(\lambda_1) + \cdots + A_n(\lambda_n)$. Setting $a_1 = \cdots = a_n = 1$ we obtain the solution before.

Inserting this solution into the third recurrence relation (3.21), we get

$$G_{(a+1)} = -LG_{(a)} + \sigma_a G_{(1)}, \tag{3.32}$$

and therefore,

$$(-1)^a G_{(a+1)} = \sum_{k=0}^{a}(-1)^k\sigma_k(\lambda^1, \dots , \lambda^n)L^{a-k}G_{(1)}, \tag{3.33}$$

which apart from an overall sign, are precisely the Killing tensors $K_{(a)}$ obtained by Benenti, Eq. (2.7).

Finally, we consider the recursion relation for the potentials V_a. First of all, we set $a = 1$, and notice that the equation

$$\mathrm{d}V_2 = (-L^* + \sigma_1)\mathrm{d}V_1,$$

implies

$$\mathrm{d}((-L^* + \sigma_1)\mathrm{d}V_1) = \mathrm{d}(G_{(1)}\mathrm{d}V_1) = 0. \tag{3.34}$$

Thus, V_1 must be a separable potential. Furthermore, we notice that the recurrence relation

$$\mathrm{d}V_{a+1} = -L^*\mathrm{d}V_a + \sigma_a \mathrm{d}V_1, \tag{3.35}$$

entails the analog of Eq. (3.31) for $\mathrm{d}V_a$,

$$(-1)^a \mathrm{d}V_{a+1} = \sum_{k=0}^{a} (-1)^k \sigma_k(\lambda^1, \ldots, \lambda^n)(L^*)^{a-k}\mathrm{d}V_1. \tag{3.36}$$

which can be simply written as

$$\mathrm{d}V_a = G_{(a)}\mathrm{d}V_1, \tag{3.37}$$

recovering again formula (2.8). Therefore, the functions V_a are the separable potentials considered by Benenti.

Remark. *It is well known that because $G_{(a)}$ are Killing tensors, then Eq. (3.34) is solvable on V_a. This solvability condition could, however, also be seen as a consequence of the Jacobi condition on the Poisson bracket P_1 entailing the existence of the Casimir function.*

To end the proof that Benenti's Hamiltonians are the coefficients of the Casimir in the Gel'fand–Zakharevich scheme, we have to show that $\hat{H}_{n+1} = 0$. In fact, in the recurrence relation Eq. (3.19), we obtain

$$\mathrm{d}f_{n+1} = [(L^*)^n - \sigma_1(L^*)^{n-1} + \cdots + (-1)^n\sigma_a]\alpha_1$$

which vanishes identically by the Cayley–Hamilton theorem. A similar statement holds for $\mathrm{d}V_{n+1}$ and $G_{(n+1)}$. Thus we can conclude that $\hat{H}_n$ is a Casimir for P_1 because

$$P_1(\mathrm{d}\hat{H}_n) = P_0(\mathrm{d}\hat{H}_{n+1}) = 0.$$

We can then complete the picture of the Casimirs of the pencil of Poisson structures P_0, P_1.

$$\begin{array}{ccccccccccccc}
 & \hat{H}_0 & & & \hat{H}_1 & & \cdots & & \hat{H}_{n-1} & & & \hat{H}_n & \\
{}^{P_0}\swarrow & & \searrow^{P_1} & {}^{P_0}\swarrow & & \searrow & & \swarrow & & \searrow^{P_1} & {}^{P_0}\swarrow & & \searrow^{P_1} \\
0 & & X_1 & & & X_2 \cdots X_{n-1} & & & & X_n & & & 0
\end{array}$$

In conclusion, we have proved the following proposition relating the geometry of the Riemannian manifold (Q, g, L) considered by Benenti to the Poisson geometry of its extended cotangent bundle $T^*Q \times \mathbf{R}$.

Theorem 1. *Let (Q,g,L) be a Riemannian manifold of the Benenti type and $V : Q \to \mathbf{R}$ a separable potential verifying the condition*

$$\mathrm{d}((L - Tr\, L)\mathrm{d}V) = 0.$$

Furthermore, let $K_{(a)}$ be the Benenti Killing tensors (2.7), V_a the iterated potentials defined by $\mathrm{d}V_a = K_{(a)}\mathrm{d}V$, and H_a the Benenti Hamiltonians (2.9). Finally, let $\hat{H}_a = H_a + \sigma_a E$ be the prolongations of the Hamiltonian H_a from T^*Q to $T^*Q \times \mathbf{R}$. Then,

1. *T^*Q endowed with the Poisson bivectors π_0 and π_1 is a bihamiltonian manifold of Poisson–Nijenhuis type.*

2. *The Hamiltonians H_j are in involution with respect to both Poisson brackets on T^*Q, and verify the cyclic recurrence relations*:

$$N^*(\mathrm{d}H_a) = \mathrm{d}H_{a+1} + \sigma_a \mathrm{d}H_1.$$

3. *$T^*Q \times \mathbf{R}$ endowed with the extended Poisson bivectors P_0 and P_1 is a bihamiltonian manifold of Gel'fand–Zakharevich type.*
4. *The Hamiltonians $\hat{H}_a$ are in involution with respect to both Poisson brackets on $T^*Q \times \mathbf{R}$ and verify Lenard's recursion relations:*

$$P_0(\mathrm{d}\hat{H}_{a+1}) = P_1(\mathrm{d}\hat{H}_a),$$

with $\hat{H}_0 = E$.
5. *The Hamiltonians $\hat{H}_a$* are the coefficients of the Casimir polynomial of the Poisson pencil on $T^*Q \times \mathbf{R}$.

From (4) and (5) it follows:
1. *The recursion relations on the Killing tensors and separable potentials given by Benenti are a particular instance of the Lenard recursion relations.*
2. *The Benenti systems (suitably extended to $T^*Q \times \mathbf{R}$ as Hamiltonian vector fields associated to the functions $\hat{H}_a$) are bihamiltonian.*

We consider this theorem a first indication of a deep relation, still to be worked out in detail, between the geometry of Killing tensors on a Riemannian manifold and the geometry of a particular class of Poisson manifolds.

3.6. Quasi-bihamiltonian systems and separability

The fact that $\hat{H}_n$ is a Casimir of the Poisson tensor P_1 has another interesting consequence. We wish to compare the geometry of T^*Q with that of $T^*Q \times \mathbf{R}$. We first notice that

$$P_1 = \pi_1 + X_{\hat{H}_1} \wedge \frac{\partial}{\partial E}.$$

Therefore,

$$0 = P_1(\mathrm{d}\hat{H}_n) = \left(\pi_1 + X_{\hat{H}_1} \wedge \frac{\partial}{\partial E}\right)(\mathrm{d}\hat{H}_n + E\mathrm{d}\sigma_n + \sigma_n \mathrm{d}E))$$
$$= \pi_1(\mathrm{d}H_n) + E\pi_1(\mathrm{d}\sigma_n) + X_{\hat{H}_1}(\mathrm{d}\hat{H}_n) \wedge \frac{\partial}{\partial E} - \sigma_n X_{\hat{H}_1}.$$

On the other hand,

$$X_{\hat{H}_1} = X_{H_1} + E X_{\sigma_1}.$$

Thus, restricting the previous equation to the symplectic submanifold $E = 0$ in $T^*Q \times \mathbf{R}$, this is restricting the previous equation to the original phase space, we obtain,

$$0 = \pi_1(\mathrm{d}H_n) + X_{H_1}(\mathrm{d}H_n) \wedge \frac{\partial}{\partial E} - \sigma_n X_{H_1}. \tag{3.38}$$

Since, $\{H_1, H_n\}_0 = 0$, Eq. (3.38) becomes

$$\pi_1(\mathrm{d}H_n) = \sigma_n \pi_0(\mathrm{d}H_1). \tag{3.39}$$

This shows that the system with Hamiltonian H_1 is Pfaffian quasi-bihamiltonian on T^*Q in the terminology of [15]. Furthermore, the previous discussion also explains why certain classes of quasi-bihamiltonian systems are separable.

4. Bihamiltonian structures and Stäckel separability

4.1. The general setting

From the previous discussion a geometrical picture emerges that contains the basic ingredients of Stäckel separability. This picture consists of a GZ bihamiltonian manifold M with Poisson brackets P_0 and P_1 of maximal rank; a symplectic submanifold S for the Poisson bracket P_0 and a vector field Z transverse to S such that

$$\mathcal{L}_Z P_1 = X \wedge Z.$$

The vector field Z can be (at least in a tubular neighborhood of S) written as $Z = \partial/\partial E$ where is E is the Casimir of P_0. Now we shall consider a Lagrangian submanifold Q of S. By Darboux–Weinstein theorem there is a neighborhood of Q on S which is symplectically equivalent to a neighborhood on the zero section of T^*Q. Thus, we can restrict our attention to such neighborhood and we can think that $S = T^*Q$ without losing generality. The Casimir E will be iterable on M and from it we will obtain a family of commuting Hamiltonians $\hat{H}_1, \dots, \hat{H}_n$ in $T^*Q \times \mathbf{R}$ that will be a generalized Stäckel family of constants of the motion for the Hamiltonian system defined by $\hat{H}_1$.

If the Poisson tensor P_1 is homogeneous of degree -1 with respect to the momentum coordinates of $T^*Q \times \mathbf{R}$ we are again in the situation described in Section 3. The family of commuting Hamiltonians will have the form given by Eq. (3.18). Moreover, the homogeneity condition implies that there exists a (1,1) tensor field L with vanishing Nijenhuis torsion such that P_1 is precisely the extension by $\hat{H}_1$ of the Poisson bracket on T^*Q defined by L.

We reproduce in this way all the results discussed so far. Notice however that the choice of the symplectic submanifold S and the Lagrangian submanifold is largely undetermined. This means that different choices of the submanifold Q can lead to families that will provide separable systems or not. We do not know in advance for which S and Q the Hamiltonians H_a are quadratic, if any. Moreover, if we find a couple $Q \subset S$ such that P_1 is homogeneous then automatically the Hamiltonians will be quadratic and of Stäckel form in these coordinates, hence the system will be orthogonal separable and eventually it could be solved completely.

In this last section we shall present two examples of this converse standpoint, without probing its theoretical basis.

4.2. The separability of stationary reductions of KdV

We shall start with a rather simple example. We shall consider the simplest stationary reduction of the KdV hierarchy. The KdV equation is given by

$$\frac{\partial u}{\partial t_3} = u_{xxx} + 6uu_x.$$

The submanifold M is defined inside the state space of the KdV hierarchy by the equation

$$u_{xxx} = -6uu_x,$$

and all its differential consequences. The manifold M is a three-dimensional manifold parametrized by u, u_x and u_{xx}. The first KdV flow,

$$\frac{\partial u}{\partial t_1} = u_x,$$

will induce the flow in M,

$$\dot{u} = u_x, \quad \dot{u}_x = u_{xx}, \quad \dot{u}_{xx} = u_{xxx} = -6uu_x.$$

We will use the variables v to denote u_x and $w = u_{xx}$ (to emphasize that presently they should be regarded as independent coordinates and not as successive derivatives of a function $u(x)$). The vector field thus obtained is given by

$$X = v\frac{\partial}{\partial u} + w\frac{\partial}{\partial v} - 6uv\frac{\partial}{\partial w}. \tag{4.40}$$

It is well-known that it is bihamiltonian. The first Poisson tensor is,

$$P_0 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 6u \\ 0 & -6u & 0 \end{bmatrix}, \tag{4.41}$$

or equivalently,

$$\{u, v\}_0 = 1, \quad \{v, w\}_0 = 6u.$$

The Hamiltonian for X is

$$\hat{H}_1 = v^2/2 - uw + 2u^3. \tag{4.42}$$

The second Poisson structure is given by the tensor P_1,

$$P_1 = \begin{bmatrix} 0 & u & v \\ -u & 0 & 6u^2 + w \\ -v & -6u^2 - w & 0 \end{bmatrix}, \tag{4.43}$$

or equivalently,

$$\{u, v\}_1 = u, \quad \{u, w\}_1 = v, \quad \{v, w\}_1 = 6u^2 + w.$$

The Hamiltonian of the vector field X with respect to the second Poisson structure is

$$E = w + 3u^2, \tag{4.44}$$

i.e., $X = P_0(\mathrm{d}\hat{H}_1) = P_1(\mathrm{d}E)$. Moreover, E is a Casimir for the Poisson tensor P_0. Hence we have obtained a bihamiltonian structure of maximal rank with the recurrence relation

$$\{., E\}_1 = \{., \hat{H}_1\}_0.$$

To prove that the bihamiltonian manifold M is of Benenti type it is convenient to use better coordinates $q, p, E, (q = u, p = v, E = w + 3u^2)$ to reconcile our notations with the previous ones. We notice that the planes E = constant, are symplectic submanifolds for P_0. Consequently, we realize that $M = T^*Q \times \mathbf{R}$, where $Q = \mathbf{R}$ is parametrized by q, and

$$P_0 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad P_1 = \begin{bmatrix} 0 & q & p \\ -q & 0 & E - 3q^2 \\ -p & -E + 3q^2 & 0 \end{bmatrix}.$$

In the new coordinates the Hamiltonian $\hat{H}_1$, Eq. (4.42), reads,

$$\hat{H}_1 = \tfrac{1}{2}p^2 + 5q^3 - qE,$$

which is quadratic. From P_1 we read the tensor field

$$L(q) = -q\frac{\partial}{\partial q} \otimes \mathrm{d}q,$$

of type (1,1) defined on Q. From the Hamiltonian $\hat{H}_1$ we can read the metric tensor

$$\mathrm{d}s^2 = \mathrm{d}q^2,$$

the separable potential

$$V(q) = 5q^3,$$

and the function

$$f(q) = -q.$$

Notice that $f = TrL$. The tensor L above provides the simplest example of a nonconstant (1,1) tensor verifying the conditions in Proposition 2. We thus conclude that the stationary submanifold M associated with the KdV hierarchy is the extended cotangent bundle $T^*Q \times \mathbf{R}$ of a Riemann manifold (Q,g,L) of Benenti type.

As a far from trivial example combining all the features that we have discussed before, we now consider a higher stationary reduction of the KdV flows. By considering the seventh-order KdV stationary reduction, we arrive to a Novikov–Dubrovin system defined on a seven-dimensional manifold M_7 with coordinates $(q_1, q_2, q_3, p_1, p_2, p_3, E)$ [5]. This system is a Gel'fand–Zakharevich bihamiltonian system with Poisson structures P_0, P_1 defined by the matrices

$$P_0 = \begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0 & -L & a \\ L^t & M & b \\ -a^t & -b^t & 0 \end{bmatrix}, \tag{4.45}$$

with

$$L = \begin{bmatrix} q_1 & -1 & 0 \\ 2q_2 & q_1 & q_3 \\ q_3 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & p_2 & 0 \\ -p_2 & 0 & -p_3 \\ 0 & p_3 & 0 \end{bmatrix}, \tag{4.46}$$

and Hamiltonian

$$\hat{H}_1 = \tfrac{1}{2}(p_1 p_2 + p_3^2) + V_1(q_1, q_2, q_3) - 2q_1 E. \tag{4.47}$$

As usual,

$$a^{\mathrm{t}} = \left(\frac{\partial \hat{H}_1}{\partial p_1}, \frac{\partial \hat{H}_1}{\partial p_2}, \frac{\partial \hat{H}_1}{\partial p_3} \right), \quad b^{\mathrm{t}} = \left(\frac{-\partial \hat{H}_1}{\partial q_1}, \frac{-\partial \hat{H}_1}{\partial q_2}, \frac{-\partial \hat{H}_1}{\partial q_3} \right),$$

in such a way to guarantee the recursion relation (3.15). In this example $M_7 = T^*Q \times \mathbf{R}$, with $Q = \mathbf{R}^3$ parametrized by (q_1, q_2, q_3). The metric tensor, the conformal Killing tensor L, the separable potential $V_1(q_1, q_2, q_3)$ and the associated potential f to the conformal factor are given by

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}q_1 \mathrm{d}q_2 + \mathrm{d}q_3^2, \\ L &= \left(q_1 \frac{\partial}{\partial q_1} + 2q_2 \frac{\partial}{\partial q_2} + q_3 \frac{\partial}{\partial q_3} \right) \otimes \mathrm{d}q_1 + \left(q_1 \frac{\partial}{\partial q_z} - \frac{\partial}{\partial q_1} \right) \otimes \mathrm{d}q_2 + q_3 \frac{\partial}{\partial q_2} \otimes \mathrm{d}q_3, \\ V_1 &= -\tfrac{5}{8} q_1^4 + \tfrac{5}{2} q_1^2 q_2 + \tfrac{1}{2} q_1 q_3^2 - \tfrac{1}{2} q_2^2, \quad f = -2q_1. \end{aligned}$$

Once again, these objects verify all the conditions of the Benenti scheme and $f = \mathrm{Tr}L$. Therefore, the stationary submanifold M_7 of the KdV hierarchy is the extended contangent bundle T^*Q of a Benenti manifold (Q, g, L).

By using the previous results we can easily compute the coefficients of the Casimir function on M_7, and therefore the remaining Killing tensors on $Q = \mathbf{R}^3$ and separable potentials. After a tedious computation we get,

$$\begin{aligned} &\hat{H}_0 = E, \quad \hat{H}_1 = \tfrac{1}{2}(p_1 p_2 + p_3^2) + V_1(q_1, q_2, q_3) - 2q_1 E, \\ &\hat{H}_2 = \tfrac{1}{2}\hat{G}_{(2)} + V_2 + (q_1^2 + 2q_2)E, \quad \hat{H}_3 = \tfrac{1}{2}\hat{G}_{(3)} + V_3 - q_3^2 E, \end{aligned}$$

where,

$$G_{(2)} = -LG_{(1)} + 2q_1 G_{(1)}, \quad G_{(3)} = L^2 G_{(1)} - 2q_1 L G_{(1)} + (q_1^2 + 2q_2) G_{(1)}, \tag{4.48}$$

and the functions V_2 and V_3 are implicitly defined by the recurrence relations,

$$\mathrm{d}V_2 = -L^* \mathrm{d}V_1 + 2q_1 \mathrm{d}V_1, \quad \mathrm{d}V_3 = [(L^*)^2 - 2q_1 L^* + (q_1^2 + 2q_2] \mathrm{d}V_1. \tag{4.49}$$

We avoid to write them in extenso. As for the Killing tensors on Q we get

$$\hat{G}_{(2)} = \tfrac{1}{2}(p_1^2 + 2q_1p_1p_2 - 2q_2p_2^2 - 2q_3p_2p_3 + 2q_1p_3^2),\ \hat{G}_{(3)} = \tfrac{1}{2}(q_3^2p_2^2 - 2q_3p_1p_3 - 2q_2q_3p_2p_3 + (q_1^2 + 2q_2)p_3^2).$$

4.3. Hydrodynamic systems

Recently it has been noted the relation between the integrability of a system of hydrodynamic type

$$q_t^i = v_j^i(q)q_x^j, \tag{4.50}$$

and the Stäckel separability of certain finite dimensional Hamiltonian systems [7].

Tsarev [20] proved Novikov's hypothesis on the integrability of systems of hydrodynamic type stating that the system (4.50) is integrable if there exist Riemann invariants for it. If we think of the functions v_j^i as the component functions of a (1,1) tensor L on the manifold Q, Novikov's condition is equivalent to the diagonalization of L in a local chart. Thus if L has vanishing Nijenhuis torsion, it will define integrable systems of hydrodynamical type. But such a tensor L will also define a Benenti system for a suitable metric g. Then considering the Hamiltonians H given by the kinetic energy of the metric g and F the second Killing tensor in the family defined by L we will obtain the Stäckel separable Hamiltonians used in [7] to integrate the system (4.50).

Moreover, the system of hydrodynamic type before will belong to a family of $n-1$ commuting flows defined by the systems of hydrodynamic type corresponding to the pairs of Hamiltonians H_1, H_a in the family of quadratic Killing tensors (2.7).

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